JOURNAL OF DIFFERENTIAL EQUATIONS 4, 399–407 (1968)

A Boundary Value Problem for a Class of Nonlinear Ordinary Differential Equations*

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Received April 12, 1967

1. INTRODUCTION

The purpose of this paper is to prove existence and uniqueness for solutions of the nonlinear ordinary differential equation

$$h(u)\left(\frac{d^2u}{dx^2}+f(x)\frac{du}{dx}\right)+1=0 \tag{1.1}$$

with boundary conditions

$$u'(0)=0, \qquad u(1)=\lambda\geqslant 0, \tag{1.2}$$

(the prime denotes differentiation with respect to x) where the functions $f(x)$ and $h(u)$ have certain properties to be specified below. The situation in which $h(u)$ has zeroes often causes difficulty. This results from the fact that solutions of the initial value problem (if they exist) may approach a zero of $h(u)$ as x approaches some value $x^* < 1$. In this case it would be impossible to continue the solution for larger values of x.

The Föppl–Hencky equation which arises in elastic membrane theory is an example of an equation of this type (cf. [*1*], [*2*]). The Föppl–Hencky equation has the form

$$u^2\left(\frac{d^2u}{dx^2}+\frac{3}{x}\frac{du}{dx}\right)+2=0, \tag{1.3}$$

where u is essentially the dimensionless radial stress which develops in a circular membrane when subjected to a constant normal pressure. In [*3*] existence and uniqueness was shown for the Föppl–Hencky equation when

* This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the U. S. Army Research Office, Durham, Contract DA-31-124-ARO-D-360.

λ [cf. (1.2)] is sufficiently large. In the present paper the results obtained in [3] will be extended to the Eq. (1.1) where

$$f(x) = \frac{c}{x} + g(x), \qquad c > -1, \tag{1.4}$$

and $g(x)$ is bounded and integrable on [0, 1]. $h(u)$ will be assumed positive and nondecreasing on some interval depending on λ, and will satisfy a Lipschitz condition on this interval.

In Section 2 the existence theorem will be proven by converting (1.1) with boundary conditions (1.2) into an equivalent nonlinear integral equation. To illustrate the application of the theorem, the Föppl–Hencky equation (1.3) will be treated, in addition to a second equation

$$K \sin u \left(\frac{d^2u}{dx^2} + \frac{c}{x}\frac{du}{dx}\right) + 1 = 0. \tag{1.5}$$

In Section 3 properties of the solution of (1.1) will be discussed and uniqueness will be shown.

2. Existence

The question of existence of solutions to the Eq. (1.1) with boundary conditions (1.2) is more easily treated if the equation is rewritten as an integral equation. For this purpose write (1.1) in the selfadjoint form

$$\frac{d}{dx}\left\{G(x)\frac{du}{dx}\right\} = -\frac{G(x)}{h(u)} \tag{2.1}$$

where

$$G(x) = x^c e^{\int_0^x g(\tau)\,d\tau}. \tag{2.2}$$

Eq. (2.1) may be integrated twice so that

$$u(x) = u(0) - \int_0^x \left\{\frac{1}{G(\omega)}\int_0^\omega \frac{G(\tau)}{h(u)}\,d\tau\right\} d\omega \tag{2.3}$$

[recall that $u'(0) = 0$]. It is easily verified that the double integral in (2.3) exists if $c > -1$. The boundary condition $u(1) = \lambda$ may be introduced into the equation by noting that (cf. (2.3)]

$$u(1) = \lambda = u(0) - \int_0^1 \left\{\frac{1}{G(\omega)}\int_0^\omega \frac{G(\tau)}{h(u)}\,d\tau\right\} d\omega.$$

Thus (2.3) takes the form

$$u(x) = \lambda + \int_x^1 \left\{ \frac{1}{G(\omega)} \int_0^\omega \frac{G(\tau)}{h(u)} \, d\tau \right\} d\omega. \tag{2.4}$$

Any solution of (1.1) with boundary conditions (1.2) satisfies the integral equation (2.4). Conversely, it may be verified that every solution of (2.4) is a solution of (1.1) with boundary conditions (1.2).

In order to simplify the notation, introduce

$$I_x(\gamma) = \int_x^1 \left\{ \frac{1}{G(\omega)} \int_0^\omega G(\tau)\gamma \, d\tau \right\} d\omega \tag{2.5}$$

and note that $I_x(\gamma)$ has the properties

(i) $I_x(\gamma_1) + I_x(\gamma_2) = I_x(\gamma_1 + \gamma_2)$,
(ii) $I_x(\gamma) \geqslant 0$ if $\gamma \geqslant 0$, and
(iii) $\max_{0 \leqslant x \leqslant 1} I_x(\gamma) = I_0(\gamma)$ if $\gamma \geqslant 0$.

In this notation the integral equation (2.4) becomes

$$u(x) = \lambda + I_x\left(\frac{1}{h(u)}\right). \tag{2.6}$$

A solution of (2.6) [and hence of (1.1) with boundary conditions (1.2)] will be constructed by choosing a suitable iteration scheme. For this purpose define

$$u_{n+1} = \lambda + I_x\left(\frac{1}{h(u_n)}\right), \tag{2.7}$$

with $u_0 = \lambda$. u_1 may be explicitly calculated from (2.7) to be

$$u_1 = \lambda + \frac{I_x(1)}{h(\lambda)} \leqslant \lambda + \frac{e^{g_M - g_m}(1 - x^2)}{2(c + 1)\, h(\lambda)} \tag{2.8}$$

where $g_M = \max_{0 \leqslant x \leqslant 1} g(x)$ and $g_m = \min_{0 \leqslant x \leqslant 1} g(x)$. In order to proceed, it is necessary to assume certain properties for $h(u)$. For the present purposes it is sufficient if

(A) $h(\lambda) > 0$,
(B) $h(u)$ is nondecreasing for $\lambda \leqslant u \leqslant \lambda + I_0(1)/h(\lambda)$,
(C) $h(u)$ satisfies a Lipschitz condition for $\lambda \leqslant u \leqslant \lambda + I_0(1)/h(\lambda)$; i.e., $| h(u_2) - h(u_1)| \leqslant L(\lambda)| u_2 - u_1 |$ if $\lambda \leqslant u_1, u_2 \leqslant \lambda + I_0(1)/h(\lambda)$.

From properties (A) and (B) it follows that the iterates u_n form a nested sequence. For example, LEMMA 2.1. $u_0 \leqslant u_2 \leqslant \cdots \leqslant u_{2n} \leqslant u_{2n+1} \leqslant \cdots \leqslant u_3 \leqslant u_1$.

Proof. The proof is by induction. Eq. (2.8) implies $u_1 - u_0 = I_x(1)/h(\lambda) \geqslant 0$ [by property (A)].

Similarly $u_2 - u_0 = I_x(1/h(u_1)) \geqslant 0$ and

$$u_1 - u_2 = I_x\left(\frac{1}{h(u_0)} - \frac{1}{h(u_1)}\right) = I_x\left(\frac{h(u_1) - h(u_0)}{h(u_1)\,h(u_0)}\right) \geqslant 0$$

follows from properties (A) and (B) and the fact that $u_1 \geqslant u_0$. Thus $u_0 \leqslant u_2 \leqslant u_1$. In the same manner

$$u_1 - u_3 = I_x\left(\frac{h(u_2) - h(u_0)}{h(u_2)\,h(u_0)}\right) \geqslant 0$$

and

$$u_3 - u_2 = I_x\left(\frac{h(u_1) - h(u_2)}{h(u_1)\,h(u_2)}\right) \geqslant 0$$

so that $u_0 \leqslant u_2 \leqslant u_3 \leqslant u_1$. The induction follows in the usual manner.

The iteration scheme (2.7) defines two sequences of functions (cf. Lemma 2.1) $\{u_{2n}\}$ and $\{u_{2n+1}\}$. Each of these sequences are bounded; i.e., $\lambda \leqslant u_{2n}$, $u_{2n+1} \leqslant \lambda + I_x(1)/h(\lambda)$ for all n, and in addition, $\{u_{2n}\}$ is a monotone increasing sequence while $\{u_{2n+1}\}$ is a monotone decreasing sequence. Thus both sequences converge to limit functions; i.e.,

$$\lim_{n\to\infty} u_{2n} = u_- \leqslant u_+ = \lim_{n\to\infty} u_{2n+1}. \tag{2.9}$$

These limit functions have the property that

LEMMA 2.2. *The functions u_- and u_+ are continuous and satisfy the integral equations*

$$u_+ = \lambda + I_x(1/h(u_-)) \tag{2.10a}$$

$$u_- = \lambda + I_x(1/h(u_+)). \tag{2.10b}$$

Proof. If the sequences $\{u_{2n}\}$ and $\{u_{2n+1}\}$ are equicontinuous, the limit functions u_- and u_+ are continuous. To show the equicontinuity of these two sequences, it suffices to show that the derivative of u_n has a uniform bound for all n. From (2.7) it is clear that

$$\frac{du_{n+1}}{dx} = -\frac{1}{G(x)}\int_0^x \frac{G(\tau)}{h(u_n)}\,d\tau$$

or

$$\left| \frac{du_{n+1}}{dx} \right| \leqslant \frac{1}{G(x)\,h(\lambda)} \int_0^x G(\tau)\,d\tau \leqslant \frac{e^{g_M - g_m}}{(c+1)\,h(\lambda)}.$$

Thus u_- and u_+ are continuous. The second part of the lemma follows if $h(u)$ has property (C). Write

$$\begin{aligned} \left| u_+ - \lambda - I_x\left(\frac{1}{h(u_-)}\right)\right| &\leqslant | u_+ - u_{2n+1} | + \left| I_x\left(\frac{h(u_-) - h(u_{2n})}{h(u_-)\,h(u_{2n})}\right)\right| \\ &\leqslant | u_+ - u_{2n+1} | + \frac{1}{h(\lambda)^2} I_x(L(\lambda)\,| u_- - u_{2n} |) \\ &\leqslant | u_+ - u_{2n+1} | + \frac{L(\lambda)\,I_0(1)}{h^2(\lambda)} \max_{0 \leqslant x \leqslant 1} | u_- - u_{2n} |. \end{aligned}$$

The right side of this inequality may be made arbitrarily small by choosing n sufficiently large. Thus u_- and u_+ satisfy (2.10a). Similarly u_- and u_+ may be shown to satisfy (2.10b).

Lemma 2.2 implies that a solution of (2.4) exists if $u_- = u_+$. To decide when this is the case, introduce a new dependent variable $\gamma = u_+ - u_-$. γ will satisfy the linear integral equation (cf. (2.10))

$$\gamma = I_x\left(\frac{h(u_+) - h(u_-)}{h(u_+)\,h(u_-)(u_+ - u_-)}\,\gamma\right) \tag{2.11}$$

or, equivalently, the linear ordinary differential equation

$$\frac{d}{dx}\left\{G(x)\frac{d\gamma}{dx}\right\} + G(x)\left\{\frac{h(u_+) - h(u_-)}{h(u_+)\,h(u_-)(u_+ - u_-)}\right\}\gamma = 0 \tag{2.12}$$

with boundary conditions

$$\gamma'(0) = \gamma(1) = 0. \tag{2.13}$$

The integral equation (2.4) will have a solution if the differential equation (2.12) with boundary condition (2.13) has only the trivial solution.

It is possible to find a sufficient condition for (2.12) to have only the trivial solution by considering a related equation

$$\frac{d}{dx}\left\{x^c e^{g_m}\frac{d\gamma}{dx}\right\} + e^{g_M}\frac{L(\lambda)}{h^2(\lambda)}\,x^c\gamma = 0, \tag{2.14}$$

where it should be noted that

$$G(x)\,\frac{h(u_+) - h(u_-)}{h(u_+)\,h(u_-)(u_+ - u_-)} \leqslant \frac{e^{g_M}L(\lambda)}{h^2(\lambda)}\,x^c \tag{2.15}$$

[cf. property (C)] and

$$G(x) \geqslant x^c e^{g_m}. \tag{2.16}$$

The inequalities (2.15) and (2.16) imply that (2.12) has only the trivial solution if (2.14) has only the trivial solution (cf. The First Sturm Comparison Theorem [*4*], pp. 228 ff.). The most general nontrivial solution of (2.14) satisfying $\gamma'(0) = 0$ is given by

$$\gamma = A \frac{J_{(c-1)/2}\left(\left[\dfrac{e^{g_M - g_m} L(\lambda)}{h^2(\lambda)}\right]^{1/2} x\right)}{x^{(c-1)/2}}$$

where $J_{(c-1)/2}$ is the Bessel function of order $(c-1)/2$. Thus (2.14) [and hence (2.12)] has only the trivial solution if

$$\left(\frac{e^{g_M - g_m} L(\lambda)}{h^2(\lambda)}\right)^{1/2} < j_{(c-1)/2,1}$$

where $j_{(c-1)/2,1}$ is the smallest root of the Bessel function of order $(c-1)/2$. It follows that a sufficient condition for the integral equation (2.4) to have a solution is for λ to satisfy the inequality

$$\frac{h^2(\lambda)}{L(\lambda)} > \frac{e^{g_M - g_m}}{j^2_{(c-1)/2,1}}. \tag{2.17}$$

The existence theorem may now be stated as

THEOREM 2.1. *The Eq.* (1.1) *with the boundary conditions* (1.2) *has a solution if* $h(u)$ *has the properties* (A), (B), *and* (C) *and* λ *satisfies the inequality* (2.17). *The solution satisfies the bounds* $\lambda \leqslant u \leqslant \lambda + I_x(1)/h(\lambda)$.

Theorem 2.1 may be extended to the case where $h = h(x, u)$ if $h(x, u)$ satisfies the properties (A), (B), and (C) for each value of x in the interval [0, 1].

As an example of the application of Theorem 2.1, consider the Föppl–Hencky equation (1.3). In this case $h(u) = u^2/2$, which is monotone increasing over the whole positive axis. To find the Lipschitz constant $L(\lambda)$ for the interval $\lambda \leqslant u \leqslant \lambda + I_0(1)/h(\lambda) = \lambda + 1/4\lambda^2$ write

$$\left|\frac{u_2{}^2}{2} - \frac{u_1{}^2}{2}\right| = \frac{1}{2}(u_1 + u_2) \mid u_2 - u_1 \mid \leqslant \left(\lambda + \frac{1}{4\lambda^2}\right) \mid u_2 - u_1 \mid.$$

Thus $L(\lambda) = \lambda + 1/4\lambda^2$. Since $u^2/2$ satisfies properties (A), (B), and (C) it only remains to find λ such that

$$\frac{\lambda^4/4}{\lambda + 1/4\lambda^2} > \frac{1}{j^2_{11}}$$

[note that $(c-1)/2 = 1$ and $g(x) \equiv 0$]. It is easily verified that this inequality is satisfied if[1]

$$\lambda^3 > \frac{2}{j_{11}^2} + \frac{1}{j_{11}}\left(\frac{4}{j_{11}^2} + 1\right)^{1/2}.$$

As a second example of the application of Theorem 2.1 consider the Eq. (1.5). It is possible to prove

COROLLARY 2.1. *For any value of λ in the interval $2n\pi < \lambda < [\frac{1}{2}(4n+1)]\,\pi$ there exists a number K^* such that the Eq.* (1.5) *with boundary conditions* (1.2) *has a solution for all $K > K^*$.*

Proof. $K \sin\lambda$ is positive if λ is in the interval $(2n\pi, [\frac{1}{2}(4n+1)]\,\pi)$. In addition, $K \sin u$ is monotone increasing for

$$\lambda \leqslant u \leqslant \lambda + I_0(1)/K \sin\lambda = \lambda + 1/[2(c+1)\,K\sin\lambda]$$

if $\lambda + 1/[2(c+1)\,K\sin\lambda] < [\frac{1}{2}(4n+1)]\,\pi$ or

$$K > \frac{1}{2(c+1)([\frac{1}{2}(4n+1)]\pi - \lambda)\sin\lambda}. \tag{2.18}$$

$K \sin u$ has a uniform Lipschitz constant; i.e., $|\,K\sin u_2 - K\sin u_1\,| \leqslant K\,|\,u_2 - u_1\,|$. Therefore $K\sin u$ satisfies properties (A), (B), and (C). By Theorem 2.1 a solution exists if

$$\frac{K^2\sin^2\lambda}{K} > \frac{1}{j^2_{(c-1)/2,1}}.$$

Thus (1.5) has a solution if

$$K > K^* = \max\left(\frac{1}{2(c+1)([\frac{1}{2}(4n+1)]\pi - \lambda)\sin\lambda},\ \frac{1}{j^2_{(c-1)/2,1}\sin^2\lambda}\right).$$

3. UNIQUENESS

In the previous section it was shown that under certain conditions there was a solution of (1.1) satisfying the boundary conditions (1.2). In this section it will be shown that the solution is unique. For this purpose it is necessary to discuss certain properties of the solution.

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[1] With a more careful analysis it is possible to show that (1.3) has a solution if $\lambda^3 \geqslant 4/j_{11}^2$ (cf. [*3*]).

LEMMA 3.1. *Let $h(u)$ be strictly increasing for $\mu_1 < u < \mu_2$ and let u_1 and u_2 be solutions of* (1.1) *corresponding to λ_1 and λ_2 with the property that $\mu_1 < u_1$, $u_2 < \mu_2$. Then $u_2 \geqslant u_1$ for $0 \leqslant x \leqslant 1$ if $\lambda_2 \geqslant \lambda_1$.*

Proof. Since u_1 and u_2 are solutions of (1.1), it follows that u_1 and u_2 are solutions of (2.3); i.e.,

$$u_j = u_j(0) - \int_0^x \left\{ \frac{1}{G(\omega)} \int_0^\omega \frac{G(\tau)}{h(u_j)} \, d\tau \right\} d\omega, \qquad j = 1, 2. \tag{3.1}$$

Subtracting the two equations (3.1) it is clear that

$$u_2 - u_1 = u_2(0) - u_1(0) + \int_0^x \left\{ \frac{1}{G(\omega)} \int_0^\omega \frac{G(\tau)}{h(u_2)\, h(u_1)} (h(u_2) - h(u_1)) \, d\tau \right\} d\omega \tag{3.2}$$

where either (i) $u_2(0) > u_1(0)$, (ii) $u_2(0) = u_1(0)$, or (iii) $u_2(0) < u_1(0)$.

The proof will be by contradiction; i.e., assume there exists x^* such that $0 \leqslant x^* \leqslant 1$ and $u_2(x^*) < u_1(x^*)$. The three possible cases are shown in Fig. 3.1. The point x' in cases (i) and (iia) is assumed to be the first inter-

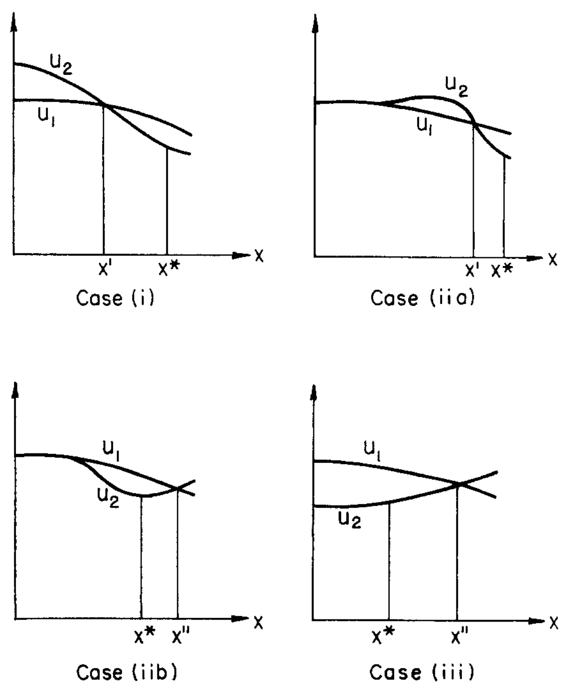

FIG. 1.

section point and note that in cases (iib) and (iii) the point x'' must exist since $u_2(1) = \lambda_2 \geqslant \lambda_1 = u_1(1)$. Cases (i) and (iia) are not possible since $u_2(0) - u_1(0) \geqslant 0$ and $h(u_2) - h(u_1) > 0$ for $0 \leqslant x < x'$. Thus (3.2) implies $u_2(x') - u_1(x') > 0$. Similarly cases (iib) and (iii) may be eliminated by noting that $u_2(0) - u_1(0) \leqslant 0$ and $h(u_2) - h(u_1) < 0$ for $0 \leqslant x < x''$ so that (3.2) implies $u_2(x'') - u_1(x'') < 0$. This is the desired contradiction.

Uniqueness is an immediate consequence of Lemma 3.1. If there are two solutions u_1 and u_2 such that $\mu_1 < u_1$, $u_2 < \mu_2$ and $u_1(1) = u_2(1) = \lambda$ Lemma 3.1 requires that $u_1 \leqslant u_2$ and $u_2 \leqslant u_1$. Thus

THEOREM 3.1. *Let $h(u)$ be strictly increasing for $\mu_1 < u < \mu_2$. There exists at most one solution u of* (1.1) *satisfying the boundary conditions* (1.2) *and having the property that $\mu_1 < u < \mu_2$.*

In conclusion it is easily shown that

THEOREM 3.2. *The solution of* (1.1) *is monotone decreasing and depends continuously on λ.*

Proof. The monotonicity of the solution is clear since (2.4) implies that

$$\frac{du}{dx} = -\frac{1}{G(x)} \int_0^x \frac{G(\tau)}{h(u)} \, d\tau < 0.$$

In order to show continuity, consider two solutions u_1 and u_2 corresponding to λ_1 and λ_2. If $\lambda_2 > \lambda_1$, Lemma 3.1 requires that $u_2 \geqslant u_1$. Thus Eq. (3.2) implies

$$\max_{0 \leqslant x \leqslant 1} (u_2(x) - u_1(x)) = u_2(1) - u_1(1) = \lambda_2 - \lambda_1$$

so that $|u_2(x) - u_1(x)| \leqslant |\lambda_2 - \lambda_1|$ for each x in $0 \leqslant x \leqslant 1$. The continuity of u as a function of λ follows.

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